CHAPTER XXVIII.

Of applying the Law of Mortality, towards reducing contingent or reversionary sums to a present value.

250. Among the useful purposes to which the law of mortality is applicable, a principal one, of daily recurrence, is that of ascertaining the present value of life-annuities and of reversionary property. The only means of such ascertainment, with a desirable correctness, are derived from a combination of the probabilities of life to be deduced from that law, with the rated interest of money; the latter being a necessary element of computation. And when proceeding on this, the importance of exclusively applying an appropriate modification of the general law of mortality, according to the class and sex of the considered lives, must now be fully understood.

251. Several branches of those results are susceptible of being tabulated by anticipation; and taking joint-lives into view, the tabulations may embrace a vast extent. On this matter, much labour has already been bestowed.

252. The celebrated Euler computed the values of single lives, from Kersseboom's tables of mortality, at *five per cent.* interest; as also Mr. de St-Cyran, from the same tables, at various rates of interest. Mr. Deparcieux likewise computed the values of single lives according to his own tables referred to the class of tontine-nominees; as also Mr. de Florencourt, from the same

tables be corrected. Mr. Dupré de St-Maur computed other tables of values, subsequently inserted in Buffon's work. After them, Mr. Milne supplied complete tables of the values of life-annuities, both on single and on joint-lives, computed according to various rates of interest and to the law he constructed from experience on the mortality of the town of Carlisle; which law has been eminently serviceable to science, at a time when the Northampton law was too indiscriminately and often very injudiciously applied. Subsequently, Mr. Davies published further elaborate tables of valued life-annuities and reversions, computed in the same manner, and according to various rates of interest, combined with the law of mortality he concluded from the experience of the "*Equitable*" life-insurance office. The late Mr. Mazeres and the late Dr. Price had previously supplied analogous tables; the former, according to the law deduced from Deparcieux's observations on select classes of lives in France; and the latter, according to his law concluded from the Northampton experience, as also to the law he concluded from Mr. Wargentin's observations on the mortality in Sweden. Other sets of tables, falling under a similar description, have lately been published by Mr. Finlaison; their computation being from a law of mortality the discussion of which has been the matter of our XX[th] chapter. And lastly, the writer of these pages has considerably advanced his tabulated valuations of life-annuities, of reversions, etc., under the respective combinations of distinct rates of interest with each of the *ten* modifications here set forth of the law of mortality: but the publication of those extensive tables must be postponed, until leisure shall have been found for their completion, if then it may be reconciled with other considerations.

253. Meanwhile it may be proper to state the following principles and methods, upon which such computations are

generally to be proceeded with; as also to point out some means of securing their accuracy, with all possibly economy of time and labour.

254. Retaining for the symbols x and y their preceding significations, — the former, that of any year of age completed by the considered individual,—and the latter, that of any quantity of survivors at such age, out of a given quantity of births, according to the decrement referable to the appropriate law of mortality; let r further signify a fractional quantity, equal to one year's interest in a single instalment, supposing *unity* to represent the principal sum; and let also X signify a term of forthcoming years, to the expiration of which will be referred any sum then receivable or payable, when the question is of reducing such future sum to a present equivalent. Then $\frac{1}{(1+r)^{X}}$ shall express a present value of the sum certainly forthcoming at the expiry of the term signified by X, that sum represented by *unity* discountable for that term at compound-interest according to its ratio r, at the same time as $-X\lambda(1+r)$ is the logarithm of that value, expressible only by a fractional quantity; and $u_{\mathrm{X}\atop x} = \frac{y_{x+\mathrm{X}}}{y_x(1+r)^{\mathrm{X}}}$ shall likewise express, by a still smaller quantity, the present value of a future and contingent sum also represented by *unity* and available at the expiry of the term X, but conditionally on the assigned life actually of the age x being still extant at the further age $x+\mathrm{X}$. That general expression $u_{\mathrm{X}\atop x}$, the logarithm of which is $\lambda y_{x+\mathrm{X}} - \lambda y_x - \mathrm{X}\lambda(1+r)$, shall be the basis on which to rest the computation of any value hereafter signified by v_x, being that of the annuity depending on a single life.

255. Having admitted X generally to represent all successive

numbers of years, beyond that of the individual's present age x; if the corresponding values represented by u be accordingly computed, we shall have a series u_{X}, correctly expressing present values of the contingent *unity* incoming at the expiration of every succeeding year, until any term at which the corresponding value becomes so reduced as no longer to be of consideration; $v = \Sigma\, u_{\mathrm{X}}$, the sum of partial values u_{X}, shall then express the total value or *Principal*, by a quantity of *years' purchase* of such annuity, or of the income in a single payment at the expiration of each further year of the considered life: understanding, however, that no ultimate claim of income is to be grounded on any period of days elapsing between the last revolved year and the failure of such life.

256. The method of computing the value of an annuity made to depend on the joint continuance of two lives is perfectly analogous with the preceding. Supposing those lives of the same class, sex and age, a series U_{X} will be formed of successively partial values $U = \frac{y^2_{x+\mathrm{X}}}{y^2_x(1+r)}$; whence $V = \Sigma' U_{\mathrm{X}}$ shall be the value of such an annuity, also expressed by a quantity of years' purchase, with the same understanding as above.

257. Supposing again those lives to be of different ages, though of similar class and sex: if then the ages are respectively signified by x and by x', the preceding formula will be converted into $U = \frac{y_{x+\mathrm{X}} \times y_{x'+\mathrm{X}}}{y_x \times y_{x'} \times (1+r)^{\mathrm{X}}}$; whence $V = \Sigma\, U_{\mathrm{X}}$ as before. But in case of the lives being of different sexes, or of different classes, it will be necessary to apply the appropriate modifications of the law of mortality; multiplying the quantity $\frac{y_{x+\mathrm{X}}}{y_x}$ which the one modification may represent, by its

corresponding quantity $\frac{y_{x'+X}}{y_{x'}}$ according to the other, and dividing the product by $(1+r)^X$; whence the process stated in the 255th paragraph shall equally yield a series U_X, the summation of which $V = \Sigma\, U_X$, exhibiting the present value of the annuity on joint-life; always with the understanding aforesaid.

258. When the question is of tabulating, by anticipation, the values either of single or of joint lives, according to successive years of age, and to any given rate of interest, a great economy of time and of mental labour, as also a security against incidental errors, will result from the following mode of proceeding.

259. Having stated in one column all logarithms of the successive quantities u_X, with reference either to any earliest age signified by x, or to the earliest two ages signified by x and by x' in the case of joint lives, those logarithms being *positive* quantities, in expressing which, all the *negative* indices shall be omitted; every succeeding column, referring to ages more advanced, will be formed by the one immediately preceding, exclusively of its first term, and adding to each of its subsequent terms an equal quantity, being the arithmetic complement of that first term of the series. The motive is obvious. Each quantity, either u or U, being produced by the probability that a single life aged x years,—or else two joint-lives as described, — shall endure another year, multiplied by the discounted value of *unity* for one year; it follows, that when such first year is expired the mere *probability* has become a *certainty*, as also that the discounted sum has recovered an integral value, as regards either the life aged x having then attained the age $x+1$, or as regards the joint-lives both of which have in

the same manner advanced another year; whence the logarithm of the probability, and that of the discounted value, shall have respectively risen to 0, from being negative quantities referred to the year of age immediately preceding; and all other logarithms in the column $\lambda\, u_{1_x}$ shall increase by a similar quantity, to form the column $\lambda\, u_{1_{x+1}}$. When thus proceeding, errors will be guarded against, by observing, as a natural consequence, that the successive, quantities $\lambda u_{1_{x+1}}$, $\lambda u_{1_{x+2}}$, $\lambda u_{1_{x+3}}$, $\lambda u_{1_{x+4}}$ ...... $\lambda u_{1_{x+n}}$, must be respectively equal to the corresponding quantities $\lambda u_{2_x} - \lambda u_{1_x}$, $\lambda u_{3_x} - \lambda u_{2_x}$, $\lambda u_{4_x} - \lambda u_{3_x}$, $\lambda u_{5_x} - \lambda u_{4_x}$ ...... $\lambda u_{n+1_x} - \lambda u_{n_x}$, in the same order of succession; whence no incidental error could escape detection. And further the chances of error will be diminished, as also a considerable saving of time will be obtained, by avoiding to appropriate in the logarithms more figures than strictly necessary. *Five* are sufficient for the purpose of all desirable accuracy, which then admits *four* decimal figures to express the most elevated among the fractional quantities u_X; and it would be a mere delusion to imagine that a nearer approach to absolute correctness, in a matter resting entirely on probabilities and approximations, could be obtainable through the introduction of any greater number of decimal figures in the expressed valuations of annuities by quantities of years' purchase. The writer derived great economy of time from using Lalande's table of logarithms thus limited, and carefully transcribed within the smallest possible space. Errors may however arise on converting the $\lambda\, u_X$ into their corresponding quantities u_X, by the tables of logarithms, although with ordinary attention they should but seldom occur when having only *four* figures to set down; as also further errors on summing up $v = \Sigma\, u_X$. But a ready method of detecting all such errors will be to note the successive differences $v_x \pm v_{x+1}$, $v_{x+1} \pm v_{x+2}$, $v_{x+2} \pm v_{x+3}$, etc., throughout the whole series of computed values; and to remark where the regular progression of those differences may be materially disturbed, thus pointing out any specific quantity v_x, or series u_x, involving probable error. All that is here mentioned, referably to u and to v, equally applies to U and to V regarding annuities on joint-life.

260. The present value, or *Principal*, of an annuity contingent on the longest of any two lives, — which implies its being payable until both lives shall have dropt, — is immediately determinable from a previous ascertainment of the respective values of three equal annuities, two of which contingent on each of those lives singly considered, and the third on their joint-continuance. Supposing the values of the single lives respectively to be v and v', as also V to be the value of their joint continuance, the value of the longest life shall obviously be, in all cases, $v + v' - V$, or the sum of the first two values abating the third, or value of the eventually shortest life; which necessarily follows from the latter's ultimately proving identical with the one or with the other of the two, whose contingencies were separately valued.

261. When stating the above rules, the life-annuities of different descriptions have been exclusively considered as accruing at the expiration of each revolved year, and as terminating with the last of those years preceding the failure of the life on which such income depended; but as other conditions may attach to the grant of any life-annuity, it is necessary further to state the modifications of value consequent on each particular condition,

differing from those first supposed and always assumed when the values are tabulated by anticipation.

262. In case of an annuity stipulated to accrue at the commencement — instead of the expiration — of every succeeding year, either of the valuations, computed as before, are to be increased by *unity* or a whole year's purchase; the only difference then consisting in the first year's anticipated income, which indeed is nothing else than an abatement on the principal sum or purchase price.

263. There are two other cases, both independent of each other, as also of the preceding; but the solution of questions involved by those cases requires previously to ascertain another quantity, which we may generally signify by t. That quantity, — either t_v, t_V, or $t_{v+v'-V}$, — expresses a *term certain* of years, during which, and by an equitable composition, the annuity of whatever description might be continued, instead of being made to depend on the uncertain duration of life. By consulting the elementary works that state invariable relations between these *five* quantities, — the *annuity*, — its *cumulated amount* at the expiry of any given term, — its value or the *principal*, — the *ratio* of interest, — and the *term* of years of the annuitys' continuance; and in which works it is further demonstrated, that any three of those quantities being stated, the other two become ascertainable; it will be found that the relative term t, according to the respectively predetermined values, is either $t_v = \frac{-\lambda(1 - rv)}{\lambda(1+r)}$, $t_V = \frac{-\lambda(1 - rV)}{\lambda(1+r)}$, or $t_{v+v'-V}$ $= \frac{-\lambda(1 - r[v + v' - V])}{\lambda(1+r)}$.

264. Reverting to the two cases mentioned in the first

clause of the preceding paragraph: if the annuity continued payable, in due proportion, for any intervening period between the last revolved year and the demise of the understood life or lives, the value, predetermined without that condition, would in this case admit an increase equal to a half-year's income discounted at compound-interest, with reference to the term t computed according to the above direction; and if the annuity accrued by *half-yearly* or by *quarterly* instalments, in corresponding proportions, instead of accruing by single and annual instalments, the increased value would then, and according to either of those proportions, be

$$\frac{(1+\frac{r}{2})^{2t}-1}{(1+r)^{t}-1}, \text{ or } \frac{(1+\frac{r}{4})^{4t}-1}{(1+r)^{t}-1}.$$

265. As those cases of exception, to the general rule for valuing life-annuities, are perfectly independant of each other, they admit such combinations as to require the computations of increase, above the simple and tabulated values, to be governed by the respective analogies; observing always, that the fractional quantity signified by r is understood invariable, when performing different functions in the original and in the supplementary computations. Hence, and supposing anticipated accretions of only the half-yearly or the quarterly instalments, the increase of the annuity's value would be no more than a fourth or an eighth part of a years' income discounted for the term t, if notwithstanding such anticipation the annuity continued proportionally payable after the last revolved half-year or quarter, until the demise; whilst, on the other hand, cumulated proportions of the increased value should be consequent on a combination of the two cases stated in the preceding paragraph, or on a combination of either with the condition of anticipated instalments; and the latter condition alone, if the instalments

were half-yearly or quarterly, would merely add a corresponding proportion to any predetermined value of the annuity.

266. As regards the present value of any reversionary property, available on the failure of a single life, — on that of either the one or the other of two lives jointly considered, — or else on the failure of both lives, — the ascertainment of such values entirely depends on that of the annuities contingent on those lives as respectively described; excluding the consideration of any one amongst the extraordinary conditions discussed in the 261st and subsequent paragraphs.

267. Now admitting a to signify the reversionary sum, or the value of any other reversionary property; and further admitting b, B, and $b + b' - \mathrm{B}$, respectively to signify the corresponding and present values of such reversions, after failure of a single life valued v,—after failure of any two lives the joint-continuance of which valued V,—or after failure of both those lives the longest of which valued $v + v' - \mathrm{V}$; there is, with reference to all cases, the following determinations of those present values:

$$b = \frac{a - arv}{1 + \frac{r}{2}};\ \mathrm{B} = \frac{a - ar\mathrm{V}}{1 + \frac{r}{2}}, \text{ and } b + b' - \mathrm{B} = \frac{a - ar(v + v') - ar\mathrm{V}}{1 + \frac{r}{2}}.$$

268. If the reversion valued B were conditional on the first failing, of the two lives, being one selected preferably to the other; the value of such conditional reversion would be, — proportionally with the amount of unconditional valuation, — as the fractional quantity expressing the probability of the selected life's surviving the other,—to *unity* or the sum of their reciprocal probabilities

of survivorship. Mr. Morgan's rule (*), for computing those probabilities, is however without application in the case of two lives differing in sex or in class, and thence requiring a reference to distinct modifications of the law of mortality.

269. Having thus ascertained the equivalent of any reversionary property, it is easy further to determine an annual sum substitutable for that equivalent and depending on the contingencies of life in the presupposed cases, of single, of joint, or of longest life; which annual sum is usually denominated *Premium of Insurance*. Such premium being nothing else than an annuity, into which the principal sum is converted by mutual agreement, and considering that it is usually discharged by anticipation of each year in a single payment; it shall then be, either $\frac{b}{v+1}$, $\frac{B}{V+1}$, or $\frac{b+b'-B}{v+v'-V+1}$, according to the specific case. But it must be understood, that any premium of life-insurance computed on those grounds, — however equitable, — leaves entirely out of consideration the charges of management, requisite profit, and liabilities necessarily incurred by public institutions, when transacting business of that nature.

270. It is immediately consequent on what has just been stated, that the title to property, denominated *Policy of Insurance on Life*, possesses a value increasing with the advance of time since the contract was entered into. At any posterior period, the real value of the reversion, whether that value were b, B, or $b+b'-B$, will be more or less superior

(*) Mr. Morgan has computed the probabilities of survivorship, according to the Sweden mortality-tables; as also Mr. Milne, according to his tables for Carlisle.

to what it was originally; whilst the annuity or premium, charged upon that original value, will represent a smaller capital, at which it should be redeemable. The property in a Policy may therefore at all times be equitably purchased at the then present value of the reversion, according to the ages of the life or lives on which it depends; abating the present value, also, of the life-annuity as then describable. This simple rule applies to all possible cases; there existing no adequate motive for excluding from the benefit of life-insurance, either joint-lives, or a survivor's life. It is of no small importance for the public, at large, to be facilitated in ascertaining the fair value of such titles of property; a due information respecting which may render the unpretending individual less dependant on arbitrary dealing. Some offices of life-insurance are understood to notify, at the time of contracting, the gradually increasing valuations from year to year, at which they are willing to purchase the policies originating with themselves: a very laudable regulation, which all offices of that description ought to adopt. It is but seldom that contingent property of any kind, when thrown on the market, may obtain its full value; and a life-annuity in particular has, from special fitness, its greatest appreciable value when in the nominee's possession.

271. This chapter is confined to stating those applications of the law of mortality which relate to results susceptible of tabulation for occasional use. There are numerous other objects of application, the appropriate rules for most of which are to be found in various publications, chiefly those of Mr. Morgan. The circumscribed purpose of the present work further renders unnecessary that those statements should embrace the joint consideration of any more than two lives, the valuation of contingencies depending on which might equally be called for.